

THE CAPELLI IDENTITY AND RADON TRANSFORM FOR GRASSMANNIANS

SIDDHARTHA SAHI AND GENKAI ZHANG

ABSTRACT. We study a family $C_{s,l}$ of Capelli-type invariant differential operators on the space of rectangular matrices over a real division algebra. The $C_{s,l}$ descend to invariant differential operators on the corresponding Grassmannian, which is a compact symmetric space, and we determine the image of the $C_{s,l}$ under the Harish-Chandra homomorphism. We also obtain analogous results for corresponding operators on the non-compact duals of the Grassmannians, and for line bundles. As an application we obtain a Radon inversion formula, which generalizes a recent result of B. Rubin for real Grassmannians.

1. INTRODUCTION

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ be a division algebra of real dimension $d = 1, 2, 4$. The first main result of this paper is a new Capelli-type identity for Grassmannians over \mathbb{F} . This generalizes earlier work of one of the authors [32], which solved a problem posed by Howe-Lee [15] for $\mathbb{F} = \mathbb{R}$. Let r be an integer such that $1 \leq r \leq n/2$, and consider the three spaces

$$(1.1) \quad W = W_{n,r} = \text{Mat}_{n \times r}(\mathbb{F}), \quad X = X_{n,r} = \text{Mat}_{n \times r}^\times(\mathbb{F}), \quad Y = Y_{n,r} = \text{Gr}_{n,r}(\mathbb{F})$$

consisting, respectively, of all \mathbb{F} -matrices of shape $n \times r$, the open subset of matrices of \mathbb{F} -rank r , and the Grassmannian of r -dimensional \mathbb{F} -subspaces of \mathbb{F}^n . We write $G_n = GL_n(\mathbb{F})$ and $K_n = U_n(\mathbb{F})$ for its maximal compact subgroup. Then G_n and G_r act naturally on the left and right of W and the actions preserve X . Moreover we have natural isomorphisms

$$(1.2) \quad Y \approx X/G_r \approx K_n/(K_r \times K_{n-r}).$$

We write $w^\dagger = \overline{w}^T$ and consider the following functions/differential operators on W

$$(1.3) \quad \Psi(w) = \det(w^\dagger w), \quad L = \partial(\Psi), \quad C_{s,l} = \Psi^{s+l} L^l \Psi^{-s},$$

where ∂ is the isomorphism between polynomials and differential operators on W induced by the pairing $\langle w_1, w_2 \rangle = \text{tr}(w_1^\dagger w_2)$. Then $C_{s,l}$ is $K_n \times G_r$ invariant and by (1.2) it can be regarded as an element of the algebra $\mathbf{D}_{K_n}(Y)$ of K_n -invariant operators on $Y = K_n/(K_r \times K_{n-r})$, which is a compact symmetric space of type BC_r .

By a result of Harish-Chandra [11] one has an algebra isomorphism

$$\eta : \mathbf{D}_{K_n}(Y) \longrightarrow \mathcal{P}(\mathfrak{a}^*)^W$$

where \mathfrak{a} is a Cartan subspace of Y , W is the restricted Weyl group, and $\mathcal{P}(\mathfrak{a}^*)^W$ is the algebra of W -invariant polynomial functions on \mathfrak{a}^* . We identify $\mathfrak{a}^* \approx \mathbb{C}^r$ by choosing a

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basis $\{e_i\}$ of \mathfrak{a}^* such that the restricted roots are $\{\pm e_i, \pm 2e_i, \pm e_i \pm e_j\}$, and we set

$$(1.4) \quad \rho_j = d(n/2 - j + 1) - 1, \quad \rho = (\rho_1, \dots, \rho_r).$$

Our first result is a formula for the image of $C_{s,l}$ under the Harish-Chandra isomorphism.

Theorem 1.1. *We have $\eta(C_{s,l}) = c_{s,l}(z)$ where*

$$(1.5) \quad c_{s,l}(z) = \prod_{i=0}^{l-1} \prod_{j=1}^r [(2s - \rho_1 + 2i)^2 - z_j^2].$$

This is proved as Theorem 3.6 below, and in sections 4, 5 we prove analogous results for the non-compact duals of the Grassmannians, and for non-spherical line bundles.

For $\mathbb{F} = \mathbb{R}$ the operator $L = \partial(\Psi)$ is the *Cayley-Laplace* operator; see [25] and the references therein. Using Theorem 1.1 one can give a new proof of several key results of [25], and extend these results to \mathbb{C}, \mathbb{H} . We intend to discuss this in a subsequent paper.

Theorem 1.1 turns out to have a beautiful application to Radon inversion. For $r \leq r'$ the Radon transforms $R : Y_{n,r} \rightarrow Y_{n,r'}$ and $R' : Y_{n,r'} \rightarrow Y_{n,r}$ are defined as follows

$$(1.6) \quad Rf(y') = \int_{y \subset y'} f(y) dy, \quad R'f(x) = \int_{y' \supset y} f(y') dy',$$

where dy, dy' are certain natural invariant measures. We now assume that

$$(1.7) \quad \text{(i) } r \leq r' \leq n - r, \quad \text{(ii) } \frac{d}{2}(r' - r) \in \mathbb{Z}.$$

Theorem 1.2. *Under the conditions (1.7) we have $(SR')R = I$ where*

$$S = \frac{1}{c_{\beta,l}(\rho)} C_{\beta,l}, \quad \beta = \frac{d}{2}(n - r'), \quad l = \frac{d}{2}(r' - r).$$

This is proved as Theorem 6.8 below.

We refer the reader to [4, 10, 11] for background on the Radon transform and its modern interpretation in the context of homogeneous spaces and integral geometry. We also refer the reader to [5], [6], [8], [9], [16], [24], [21] for other examples of Radon inversion formulas, and to [1], [2], [22], [40] for the closely related cosine transform. For $\mathbb{F} = \mathbb{R}$, Rubin [26, Th 8.2] has recently obtained an inversion formula for the Funk transform on Stiefel manifolds, which is essentially equivalent to Theorem 1.2, as explained in Remark 6.9 below. Thus our result constitutes a generalization of Rubin's result.

We refer the reader to [13, 14], [19, 20] for background and perspective on the Capelli identity, to [27], [28] for applications to unitary representations, and to [17], [31] for the connection with Jack polynomials and Macdonald polynomials. The papers [19, 20] emphasize the point of view that the classical Capelli identity can be understood as the computation of the eigenvalues of a certain invariant differential operator, or equivalently the computation of its Harish-Chandra image. This is the perspective adopted in [29, 32] and also in the present paper.

The paper is organized as follows. In section 2 we introduce the necessary background, including a precise description of the differential operators $C_{s,l}$ and the module structure of the space \mathcal{R} of G_r -invariant and K -finite functions on X . In section 3 we prove our first main result, Theorem 3.6. This result is extended in sections 4 and 5 to the non-compact and non-spherical settings, respectively. Finally in section 6 we recall some basic facts about the Radon transform and prove our second main result, Theorem 6.8.

We now discuss briefly the main ideas behind the proofs of our results.

To compute the Harish-Chandra image $c_{s,l} = \eta(C_{s,l})$, we follow the approach of Kostant-Sahi [19], [20], [29]. Let \mathcal{R} be the algebra of K_n -finite functions on $Y = K_n / (K_r \times K_{n-r})$. The irreducible K_n -submodules (K_n -types) of \mathcal{R} occur with multiplicity 1, and are uniquely determined by their highest weights $\mu \in \mathfrak{a}^* \approx \mathbb{C}^r$. On each such K_n -type the operator $C_{s,l}$ acts by the scalar $c_{s,l}(\mu + \rho)$, where ρ is the half sum of positive roots. The operator $C_s = C_{s,1}$ vanishes on certain K_n -types and we show that these vanishing conditions suffice to *characterize* the polynomial $c_s = c_{s,1}$ up to an overall scalar, which is then determined by an auxiliary computation. This proves Theorem 3.6 for $C_{s,1}$ and the result for $C_{s,l}$ follows by a factorization argument.

We consider also the non-compact symmetric space D dual to the Grassmannian Y and study corresponding operators $\tilde{C}_{s,l}$. The space D can be realized as an open domain in Y . The eigenvalues of $\tilde{C}_{s,l}$ on the Harish-Chandra spherical functions can be found by using a well-known observation [11, 12] that the spherical polynomials on Y restricted to D are the Harish-Chandra functions. When Y and D are Hermitian symmetric compact or non-compact spaces, the operators $\tilde{C}_{s,l}$ also act on sections of homogeneous line bundles, and we find the corresponding eigenvalues.

The main idea behind the proof of Theorem 6.8 is straightforward – one simply compares the eigenvalues of $R'R$ to those of $C_{s,l}$. The eigenvalues of $R'R$ are computed explicitly in [8] for $\mathbb{F} = \mathbb{C}$ and stated without proof for \mathbb{R}, \mathbb{H} . However the statement in [8, Theorem 6.2] for \mathbb{H} is not correct, and it was not clear to us how to extend the ideas of [8] to obtain the right formula. Thus in this paper we give a completely different argument using ideas from [40, 41], where the Radon transform is related to the spherical transform of the sine functions on Grassmannians, and its eigenvalues are related to special values of spherical polynomials. The results of [40, 41] are proved under certain restrictions on r, n and involve certain undetermined constants. In this paper we complete these arguments by relaxing the restrictions and explicitly determining the relevant constants. This leads to the proof of Theorem 6.8.

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2. MATRIX SPACES AND GRASSMANNIANS

2.1. Matrix spaces and Jordan algebras. For $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, let $A = A_r(\mathbb{F})$ denote the space $r \times r$ Hermitian \mathbb{F} -matrices

$$A = \{u \in \text{Mat}_{r \times r}(\mathbb{F}) \mid u^\dagger = u\}.$$

Then A is a Euclidean Jordan algebra, and we write e for the identity element and tr and \det for the trace form and the Jordan norm polynomial. We refer the reader to [3] for basic results and terminology related to Jordan algebras.

Lemma 2.1. *For $u \in A$ let D_u denote the directional derivative, then we have*

$$(2.1) \quad (D_u \det)(e) = \text{tr}(u).$$

Proof. See [3, Proposition III.4.2]. □

The structure group of A is $G_r = GL_r(\mathbb{F})$, which acts on A by $g : u \mapsto gug^\dagger$. The automorphism group is $K_r = U(r, \mathbb{F})$, which is the stabilizer of e . The structure group preserves the determinant up to a scalar multiple; thus we have

$$(2.2) \quad \det(gug^\dagger) = \nu(g) \det(u),$$

where ν is a certain character of G_r .

Let $W = Mat_{n \times r}(\mathbb{F})$ be the space of $n \times r$ matrices, where we assume as before that

$$r \leq n - r.$$

We have a natural map $Q : W \rightarrow A$,

$$Q(w) = w^\dagger w.$$

By polarization we get a positive definite inner product $\langle \cdot, \cdot \rangle$ on W satisfying

$$\langle w, w \rangle = \text{tr}(w^\dagger w),$$

the trace being computed for real linear transformations. This gives us an isomorphism between polynomials and constant coefficient differential operators

$$(2.3) \quad \partial : \mathcal{P}(W) \approx \mathcal{D}(W)$$

Let $X \subset W$ be the open subset of matrices of rank r , then $Q(X)$ is the positive cone of A . Moreover if

$$(2.4) \quad x_0 = \begin{bmatrix} I_r \\ 0 \end{bmatrix}$$

then $x_0 \in X$ and $Q(x_0) = e$. The group $G_n \times G_r$ acts naturally on W , and the polynomial

$$(2.5) \quad \Psi(w) = \det Q(w) = \det(w^\dagger w)$$

transforms under $K_n \times G_r$ as follows:

$$(2.6) \quad \Psi(kwg) = \det(g^\dagger w^\dagger k^\dagger kwg) = \det(g^\dagger w^\dagger wg) = \nu(g^\dagger) \det(w^\dagger w) = \nu(g) \Psi(w).$$

2.2. Grassmannians and invariant differential operators. The Grassmannian $Y = Gr_{n,r}(\mathbb{F})$, consisting of r dimensional subspaces of \mathbb{F}^n , has several different realizations that will play a role below. First, we have a $G = G_n$ equivariant map

$$(2.7) \quad \text{col} : X \rightarrow Y,$$

where $\text{col}(x)$ is the column space of x . This descends to a homeomorphism

$$(2.8) \quad X/G_r \approx Y,$$

which realizes X as the principal G_r -bundle associated to the tautological bundle on Y . Also, the group $K = K_n$ acts transitively on Y and the stabilizer in K of

$$y_0 = \text{col}(x_0)$$

is the symmetric subgroup $M = K_r \times K_{n-r}$. This realizes Y as compact symmetric space

$$Y = K/M.$$

Finally, the stabilizer of y_0 in G is a maximal parabolic subgroup P , thus we get

$$(2.9) \quad Y = G/P.$$

This realization is of importance when studying principal series representations of G , [30, 37].

Let $C^\infty(X)^{G_r}$ be the space of G_r -invariant smooth functions on X ; by (2.8) we get a G -equivariant isomorphism

$$C^\infty(X)^{G_r} \approx C^\infty(Y).$$

More generally if ν is as in (2.2) and $s \in \mathbb{C}$ we can consider the space of G_r -equivariant functions

$$C^\infty(X)^{G_r, s} = \{f \in C^\infty(X) : f(xg) = \nu(g)^s f(x)\}.$$

Note that $\Psi(x) = \det(x^\dagger x)$ is positive on X and hence Ψ^s is a well-defined function on X . Now (2.6) implies that multiplication by Ψ^s gives a K_n -equivariant isomorphism

$$f \mapsto \Psi^s f : C^\infty(X)^{G_r, s} \approx C^\infty(X)^{G_r} \approx C^\infty(Y).$$

We now introduce the differential operators which play a key role in this paper.

Definition 2.2. With ∂ as in (2.3), $s \in \mathbb{C}$, and $l \in \mathbb{N}$, we define

$$(2.10) \quad L = \partial(\Psi), \quad C_{s,l} = \Psi^{s+l} L^l \Psi^{-s}, \quad C_s = C_{s,1}$$

Clearly $C_{s,l}$ is a $K \times G_r$ -invariant differential operator on $C^\infty(X)$. Now by a standard argument [11, Chapt. II, Sect. 3] applied to (2.8), we conclude that $C_{s,l}$ defines a K -invariant differential operator on Y

$$C_{s,l} : C^\infty(Y) \rightarrow C^\infty(Y).$$

More generally if t is another complex number, then

$$C_{s,l} : C^\infty(X)^{G_r, t} \rightarrow C^\infty(X)^{G_r, t}.$$

2.3. K -types on the Grassmannian. To study the action of the operators $C_{s,l}$ it is convenient to pass to an algebraic setting. We write \mathcal{P} for the algebra of polynomial functions on W , and for each positive integer m we define

$$\mathcal{P}^m = C^\infty(X)^{G_r, m} \cap \mathcal{P}.$$

Also let \mathcal{R} be the subspace of $C^\infty(X)^{G_r}$ consisting of K -finite functions. The operators $C_{s,l}$ preserve the spaces \mathcal{R} and \mathcal{P}^m , and we now describe their K -module structures.

As noted above, the Grassmannian $Y = K/M$ is a compact symmetric space, and we fix some notation relevant to this structure. Let \mathfrak{k} and \mathfrak{m} denote the complexified Lie algebras of $K = K_n$ and $M = K_r \times K_{n-r}$, and fix a Cartan decomposition and Cartan subalgebra

$$(2.11) \quad \mathfrak{k} = \mathfrak{m} + \mathfrak{p}, \quad \mathfrak{h} = \mathfrak{t} + \mathfrak{a}$$

where $\mathfrak{a} \subset \mathfrak{p}$ is a Cartan subspace and \mathfrak{t} is the centralizer of \mathfrak{a} in \mathfrak{m} . Since $r \leq n - r$ by assumption, the restricted root system $\Sigma(\mathfrak{a}, \mathfrak{k})$ of type BC_r . We fix a basis $\{e_i\}$ of \mathfrak{a}^* such that the positive restricted roots α and their multiplicities m_α are given in terms of $d = \dim \mathbb{F}$ as follows

α	e_i	$2e_i$	$e_i \pm e_j$
m_α	$d(n - 2r)$	$d - 1$	d

.

The half sum of the positive roots is $\rho = \sum_{j=1}^r \rho_j e_j$ where

$$(2.12) \quad \rho_j = \frac{1}{2} [2d(r - j) + 2(d - 1) + d(n - 2r)] = d(n/2 - j + 1) - 1.$$

Let Λ be the set of even partitions of length $\leq r$:

$$(2.13) \quad \Lambda = \{(\mu_1, \dots, \mu_r) \in (2\mathbb{Z})^r : \mu_1 \geq \dots \geq \mu_r \geq 0\}.$$

Then Λ parametrizes the set $(K/M)^\wedge$ of equivalence classes of irreducible M -spherical representations of K as follows. We identify $\mu \in \Lambda$ with the \mathfrak{h} -weight that vanishes on \mathfrak{t} and restricts to $\sum \mu_j e_j$ on \mathfrak{a} . For $\mathbb{F} = \mathbb{C}$ and \mathbb{H} the group $K = U(n, \mathbb{F})$ is connected, and we write V_μ for the irreducible K -module with highest weight $\mu \in \Lambda$. If $\mathbb{F} = \mathbb{R}$ then K is the disconnected group $O(n, \mathbb{R})$ and we let \tilde{V}_μ denote the irreducible representation of $K_o = SO(n, \mathbb{R})$ with highest weight μ . Then \tilde{V}_μ extends uniquely to an M -spherical representation V_μ of K ; however if $r = n - r$ then we define

$$V_\mu = \text{ind}_{K_o}^K (\tilde{V}_\mu).$$

By the Cartan-Helgason theorem, the map $\mu \mapsto V_\mu$ gives a bijection between Λ and $(K/M)^\wedge$. The following result is standard.

Lemma 2.3. *As a K -module, the algebra \mathcal{R} admits a multiplicity-free direct sum decomposition $\mathcal{R} = \bigoplus_{\mu \in \Lambda} \mathcal{R}_\mu$ where $\mathcal{R}_\mu \approx V_\mu$.*

We need a similar description of \mathcal{P}^m . For this we define

$$(2.14) \quad \Lambda^m = \{\mu \in \Lambda \mid \mu_1 \leq 2m\}.$$

Lemma 2.4. *We have a decomposition $\mathcal{P}^m = \bigoplus_{\mu \in \Lambda^m} \mathcal{P}_\mu^m$, where $\mathcal{P}_\mu^m \approx V_\mu$.*

Proof. This is proved in [40, Proposition 6.4] for $\mathbb{F} = \mathbb{R}$ and \mathbb{H} and in [38, Theorem 5.3] for $\mathbb{F} = \mathbb{C}$. It also follows from [20]. \square

3. THE CAPELLI IDENTITY FOR GRASSMANNIANS

In this section we find the spectrum of the operator $C_{s,l}$ on \mathcal{R} , the main result being Theorem 3.6 below. The key computation involves the special case $l = 1$. Thus we consider the operator

$$C_s := C_{s,1} = \Psi^{s+1} L \Psi^{-s}.$$

and we write $c_{s,\mu}$ for its eigenvalue on \mathcal{R}_μ . We first prove that $c_{s,\mu}$ is a polynomial in s and μ , and we determine its leading term in s . For any μ we denote its ρ -shift by $\bar{\mu} = \mu + \rho$. Also let \mathcal{W}_r denote the Weyl group of type BC_r acting on polynomials in r variables by sign changes and permutations.

Lemma 3.1. *There exists a polynomial $p(t, z) = p(t, z_1, \dots, z_r)$ such that $c_{s,\mu} = p(s, \bar{\mu})$. Moreover p has t -degree $\leq 2r$, total z -degree $\leq 2r$, and is \mathcal{W}_r -invariant in z .*

Proof. Fix ϕ in $\mathcal{H}(\mu)$ such that $\phi(y_0) = 1$. Viewed as a smooth function on X , we have $\phi(x_0) = 1$. Evaluating the eigenvalue equation $C_s \phi = c_{s,\mu} \phi$ at x_0 leads to

$$c_{s,\mu} = C_s \phi(x_0) = \Psi^{1-s} L(\Psi^s \phi)(x_0) = L(\Psi^s \phi)(x_0).$$

Since L is a polynomial differential operator of degree $2r$ and Ψ^s is a power of the polynomial function Ψ , we see by Leibniz rule that $c_{s,\mu}$ is a polynomial of degree $\leq 2r$ in s . The coefficient κ_j of s^j depends only on μ , or equivalently on $\bar{\mu}$, and so we can write

$$(3.1) \quad c_{s,\mu} = \sum_{i \leq 2r} \kappa_i(\bar{\mu}) s^i$$

Consider the equations (3.1) for $2r$ distinct values of s , say $s = 1, \dots, 2r$. By non-vanishing of the Vandermonde determinant, we can invert this system of equations to obtain $\gamma_{ij} \in \mathbb{Q}$ such that

$$\kappa_i(\bar{\mu}) = \sum_{j \leq 2r} \gamma_{ij} c_{j,\mu}$$

for all i and all μ . By the Harish-Chandra homomorphism we can write $c_{j,\mu} = p_j(\bar{\mu})$, where $p_j(z)$ is a \mathcal{W}_r -invariant polynomial of total degree $\leq 2r$. Then the polynomials

$$k_i(z) = \sum_{j \leq 2r} \gamma_{ij} p_j(z)$$

satisfy the same properties, and $p(t, z) = \sum_{i \leq 2r} k_i(z) t^i$ satisfies the requirements of the lemma. \square

Lemma 3.2. *The leading term of $p(t, z)$ as a polynomial in t is $2^{2r} t^{2r}$.*

Proof. First, it follows by chain rule and (2.1) that

$$(D_w \det(x^\dagger x))(x_0) = \text{tr}(x_0^\dagger w + w^\dagger x_0) = 2(x_0, w).$$

The result now follows from [32, Lemma 6.4]. \square

We now prove a simple vanishing condition for the eigenvalues.

Lemma 3.3. *If $\mu \in \Lambda$ and $m = \mu_1/2$ then $c_{-m,\mu} = 0$.*

Proof. The operator $J_m : f \mapsto \Psi^{-m} f$ maps \mathcal{P}_μ^m isomorphically onto \mathcal{R}_μ . Thus it suffices to prove that $C_{-m} J_m$ acts by 0 on \mathcal{P}_μ^m . To see this we note that

$$C_{-m} J_m = (\Psi^{-m+1} L \Psi^m) \Psi^{-m} = \Psi^{-m+1} L.$$

Now L maps \mathcal{P}_μ^m to \mathcal{P}_μ^{m-1} , which is 0 by Lemma 2.4. \square

We deduce a divisibility property of $p(t, z)$.

Lemma 3.4. *The polynomial $p(t, z)$ is divisible by $2t - \rho_1 + z_1$.*

Proof. It is enough to show that $p(t, z)$ vanishes on the hyperplane

$$2t - \rho_1 + z_1 = 0,$$

i.e. that $g(z) = p\left(\frac{\rho_1 - z_1}{2}, z\right)$ is identically 0. For $z = \bar{\mu}$ by the previous lemma we get

$$g(\bar{\mu}) = p(-\mu_1/2, \bar{\mu}) = c_{-\mu_1/2, \mu} = 0$$

Since the set $\Lambda + \rho$ is Zariski dense, the polynomial g must be identically 0. \square

We now prove an explicit formula for $p(t, z)$. Define

$$(3.2) \quad q(t, z) = \prod_{j=1}^r [t^2 - z_j^2].$$

Theorem 3.5. *We have $p(t, z) = q(2t - \rho_1, z)$.*

Proof. By the previous lemma $p(t, z)$ is divisible by $(2t - \rho_1) + z_1$, and hence by \mathcal{W}_r -invariance it is divisible by all factors of the form $(2t - \rho_1) \pm z_j$. In other words $p(t, z)$ is divisible by $q(2t - \rho_1, z)$. However by (3.2) and Lemma 3.2 both polynomials have the same leading t -coefficient $2^{2r} t^{2r}$. Therefore they must be equal. \square

For each positive integer l let $q_l(t, z)$ denote the polynomial

$$q_l(t, z) = \prod_{i=0}^{l-1} q(t + 2i, z) = \prod_{i=0}^{l-1} \prod_{j=1}^r [(t + 2i)^2 - z_j^2].$$

The main result of this section is the following.

Theorem 3.6. *The eigenvalue of $C_{s,l}$ on \mathcal{R}_μ is $q_l(2s - \rho_1, \bar{\mu})$.*

Proof. We can factorize $C_{s,l} = \Psi^{s+l} L^l \Psi^{-s}$ iteratively as follows:

$$C_{s,l} = (\Psi^{s+l} L \Psi^{-s-l+1}) (\Psi^{s+l-1} L^{l-1} \Psi^{-s}) = C_{s+l-1} C_{s,l-1} = \prod_{i=0}^{l-1} C_{s+i}.$$

Therefore by Theorem 3.5 the eigenvalue of $C_{s,l}$ on \mathcal{R}_μ is

$$\prod_{i=0}^{l-1} p(s + i, \bar{\mu}) = \prod_{i=0}^{l-1} q(2s + 2i - \rho_1, \bar{\mu}) = q_l(2s - \rho_1, \bar{\mu}).$$

□

4. NON-COMPACT GRASSMANNIANS

In this section we explain how to extend the previous results to the non-compact duals of the Grassmannians. Let J be the Hermitian form of signature $(r, n - r)$ on \mathbb{F}^n given by

$$(v, w)_J = v^\dagger J w, \quad J = \text{diag}(I_{n-r}, -I_r),$$

and let \tilde{K} be its isometry group. Thus for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ we have, respectively,

$$\tilde{K} = O(r, n - r), U(r, n - r), Sp(r, n - r).$$

Let $M = K_r \times K_{n-r}$ be as before; then M is a symmetric subgroup of \tilde{K} and the symmetric space $\tilde{Y} = \tilde{K}/M$ is the non-compact dual of the Grassmannian $Y = K/M$.

Let X be the space of $n \times r$ matrices of rank r as before, and let \tilde{X} be the open subset

$$(4.1) \quad \tilde{X} = \{z \in X : (\cdot, \cdot)_J \text{ is positive definite on } \text{col}(z)\}.$$

Assume $n - r \geq r$ as before, and consider the following differential operator on \tilde{X} ,

$$\tilde{C}_{s,l} = \tilde{\Psi}^{l+s} \tilde{L}^l \Psi^{-s}, \quad \tilde{\Psi} = \det(z^\dagger J z), \tilde{L} = \partial(\tilde{\Psi}).$$

Now \tilde{Y} can be realized as the quotient \tilde{X}/G_r , analogous to the realization X/G_r of Y (2.7)-(2.8). Thus we can identify $C^\infty(\tilde{Y})$ with the space $C^\infty(\tilde{X})^{G_r}$ of right G_r -invariant functions on \tilde{X} and $\tilde{C}_{s,l}$ then descends to a \tilde{K} -invariant differential operator on \tilde{Y} . We explain how to compute its spectrum.

The complexifications of \tilde{K} and K are conjugate inside $G_{\mathbb{C}} = GL_n(\mathbb{F})_{\mathbb{C}}$; we have

$$\text{ad}(\sigma) : K_{\mathbb{C}} \approx \tilde{K}_{\mathbb{C}}, \quad \sigma = \text{diag}(I_r, \sqrt{-1}I_{n-r}),$$

which follows by noting that $\sigma = \sigma^\dagger$ and $\sigma^\dagger J \sigma = I$. Moreover $\text{ad}(\sigma)$ fixes M , thus allowing us to transfer the structure theoretic results (2.11), (2.12) etc. from \mathfrak{k} to $\tilde{\mathfrak{k}}$. In particular $\tilde{\mathfrak{a}} = \text{ad}(\sigma) \mathfrak{a}$ is a Cartan subspace for \tilde{K}/M , and $\tilde{\mathfrak{a}}^*$ is the space of Satake parameters for M -spherical functions on \tilde{K} . For $\lambda \in \mathfrak{a}^*$ we define $\tilde{\lambda} \in \tilde{\mathfrak{a}}^*$ by

$$(4.2) \quad \tilde{\lambda} = (\lambda + \rho) \circ \text{ad}(\sigma^{-1})$$

and let $\phi_{\tilde{\lambda}} \in C^\infty(\tilde{Y})$ be the corresponding spherical function. Also as before put

$$q_l(t, z) = \prod_{i=0}^{l-1} \prod_{j=1}^r [(t + 2i)^2 - z_j^2]$$

Theorem 4.1. *For all $\lambda \in \mathfrak{a}^*$ we have $\tilde{C}_{s,l}(\phi_{\tilde{\lambda}}) = q_l(2s - \rho_1, \lambda + \rho)\phi_{\tilde{\lambda}}$.*

Proof. By the Harish-Chandra homomorphism there exists a \mathcal{W}_r -invariant polynomial p on \mathfrak{a}^* such that we have $\tilde{C}_{s,l}(\phi_{\tilde{\lambda}}) = p(\lambda + \rho)\phi_{\tilde{\lambda}}$ for all $\lambda \in \mathfrak{a}^*$. By Zariski density p is uniquely determined by its values on the set $\Lambda + \rho$, and thus it suffices to prove the theorem for $\mu \in \Lambda$.

Let \mathcal{R} be the algebra of K -finite functions in $C^\infty(X)^{G_r}$ and let $\tilde{\mathcal{R}}$ be the algebra of \tilde{K} -finite functions in $C^\infty(\tilde{X})^{G_r}$. Also let $\mathcal{Q}, \tilde{\mathcal{Q}}$ be the algebras of G_r -invariant meromorphic rational functions on the complexification $W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$, which are $K_{\mathbb{C}}$ -finite and $\tilde{K}_{\mathbb{C}}$ -finite respectively. By Lemmas 2.3 and 2.4 each element of \mathcal{R} is a rational function on W , and hence extends uniquely to a meromorphic rational function on $W_{\mathbb{C}}$. This gives us the first map in the diagram below

$$\mathcal{R} \longrightarrow \mathcal{Q} \xrightarrow{\sigma} \tilde{\mathcal{Q}} \longrightarrow \tilde{\mathcal{R}},$$

the middle map is obtained by the natural holomorphic action of $\sigma \in G_{\mathbb{C}}$ on $W_{\mathbb{C}}$, while the last is the restriction map to \tilde{X} . All maps are injective and applying them to the irreducible K -submodule $\mathcal{R}_{\mu} \subset \mathcal{R}$ as in Lemma 2.3, we obtain an irreducible finite-dimensional \tilde{K} -submodule $\tilde{\mathcal{R}}_{\mu}$ whose M -fixed vector can be identified with $\phi_{\tilde{\mu}}$.

The polynomials Ψ and $\tilde{\Psi}$ extend to holomorphic polynomials on $W_{\mathbb{C}}$, while $C_{s,l}$ and $\tilde{C}_{s,l}$ extend to holomorphic differential operators on $W_{\mathbb{C}}$. The action of σ on $W_{\mathbb{C}}$ carries Ψ to $\tilde{\Psi}$ and $C_{s,l}$ to $\tilde{C}_{s,l}$. It follows that σ intertwines the action of $C_{s,l}$ on \mathcal{R}_{μ} with the action of $\tilde{C}_{s,l}$ on $\tilde{\mathcal{R}}_{\mu}$. Thus by Theorem 3.6 $\tilde{C}_{s,l}$ acts on $\tilde{\mathcal{R}}_{\mu}$, and in particular on $\phi_{\tilde{\mu}}$, by the scalar $q_l(2s - \rho_1, \mu + \rho)$. \square

5. NONSPHERICAL CASES

We now extend the previous results to non-spherical line bundles over the Grassmannian $Y = Gr_{n,r}(\mathbb{F})$. For $\mathbb{F} = \mathbb{R}$ the nonspherical bundles are obtained by twisting by the sign character¹, and have already been treated in [32]. For $\mathbb{F} = \mathbb{H}$ there are no such bundles, therefore it remains only to consider $\mathbb{F} = \mathbb{C}$.

The Grassmannian for $\mathbb{F} = \mathbb{C}$ is the compact Hermitian symmetric space

$$Y = Gr_{n,r}(\mathbb{C}) = X/G_r = K/M = U(n)/U(r) \times U(n-r).$$

For each integer p , the character $g \mapsto \det(g)^p$ of $G_r = GL_r(\mathbb{C})$ induces a holomorphic line bundle on $Y = X/G_r$, whose space of *smooth* sections is given as follows

$$C^\infty(Y, p) = \{f \in C^\infty(X) : f(xg) = \det(g)^p f(x) \text{ for all } g \in G_r\}.$$

As a K -module we have

$$C^\infty(Y, p) = \{f \in C^\infty(K) : f(km) = \chi_p(m) f(k) \text{ for all } m \in M\}$$

where χ_p is the character of $M = U(r) \times U(n-r)$ given by $\chi_p(m_1, m_2) = \det(m_1)^p$.

¹For $r = 2$ the Grassmannian $Gr_{n,2}(\mathbb{R})$ does have a complex structure and corresponding holomorphic line bundles, but this case does not seem to fit into our framework and we shall not treat it here.

The irreducible K -submodules of $C^\infty(Y, p)$ occur with multiplicity 1 and parametrized explicitly in [33]. For Λ, Λ^m as in (2.13, 2.14) and for an integer $b \geq 0$ we define

$$(5.1) \quad \Lambda_b = \{\lambda + (b, \dots, b) \mid \lambda \in \Lambda\}, \quad \Lambda_b^m = \{\lambda + (b, \dots, b) \mid \lambda \in \Lambda^m\},$$

regard as subsets of \mathfrak{a}^* as before. Then one has the following result [33, Th 7.2].

Lemma 5.1. *For each μ in $\Lambda_{|p|}$ there is a unique K -type V_μ in $C^\infty(Y, p)$ whose highest weight restricts to μ on \mathfrak{a} .*

One also has an exact analog of the Harish-Chandra homomorphism [36, 34].

Lemma 5.2. *Let D be a K -invariant differential operator on $C^\infty(Y, p)$. There is a unique \mathcal{W}_r -invariant polynomial p_D of degree $\text{ord}(D)$ such that the eigenvalue of D on V_μ is $p_D(\mu + \rho)$.*

In our setting, with $\Psi(w) = \det(w^\dagger w)$ and $L = \partial(\Psi)$ as in (2.5), the operator

$$C_{s,l} = \Psi^{l+s} L^l \Psi^{-s}$$

descends to a K -invariant differential operator of order lr on $C^\infty(Y, p)$. We will show that its eigenvalues can be expressed in terms of the \mathcal{W}_r -invariant polynomial

$$q_l(t, z) = \prod_{i=0}^{l-1} \prod_{j=1}^r [(t + 2i)^2 - z_j^2].$$

We first construct an analog of the space \mathcal{P}^m . Let \mathcal{R}^m be the space of holomorphic polynomials f on $W = \text{Mat}_{n \times r}(\mathbb{C})$ satisfying

$$(5.2) \quad f(wg) = \det(g)^m f(w) \text{ for } g \in G_r,$$

and let $\mathcal{R}^{m+b,m}$ be the space of sesqui-holomorphic polynomials f on $W \times \overline{W}$ satisfying

$$f(w_1 g_1, w_2 g_2) = \det(g_1)^{m+b} \overline{\det(g_2)^m} f(w_1, w_2) \text{ for } g_1, g_2 \in G_r.$$

We will describe the K -type structure of $\mathcal{R}^{m+b,m}$ following [38]. We consider two subgroups of $G = GL(n, \mathbb{C})$, written as $(r, n-r) \times (r, n-r)$ block matrices, and a character

$$P = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \right\}, \quad Q = \left\{ \begin{bmatrix} 1 & a_{12} \\ 0 & a_{22} \end{bmatrix} \right\}, \quad \chi_m \left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \right) = \det(a_{11})^m.$$

Then Q is the stabilizer of $x_0 \in X$ (2.4) and so we have natural isomorphisms

$$X \approx G/Q, \quad Y \approx X/G_r \approx G/P.$$

Thus if \mathcal{L}_m is the line bundle on G/P induced by the character χ_m then polynomials in \mathcal{R}^m , after restriction to the open set X , can be identified as holomorphic sections of \mathcal{L}_m .

We realize $C^\infty(Y, b)$ as the space of smooth functions on X transforming as $f(xg) = \det^b(g) f(x)$, $g \in G_r$.

Lemma 5.3. *If $b \geq 0$ then the map $J_m : f(x, y) \mapsto \det(x^\dagger x)^{-m} f(x, x)$ defines a K -isomorphism from $\mathcal{R}^{m+b,m}$ into a subspace of $C^\infty(Y, b)$ and we have $\mathcal{R}^{m+b,m} \approx \sum_{\mu \in \Lambda_b^m} V_\mu$ under the diagonal action of K .*

Proof. It is easy to see that the map $f \otimes g \mapsto F$, $F(w_1, w_2) = f(w_1)g(w_2)$ is a K -equivariant isomorphism from $\mathcal{R}^{m+b} \otimes \overline{\mathcal{R}^m}$ to $\mathcal{R}^{m+b,m}$. The result follows from the decomposition of the tensor product in [38, Theorem 5.3]. (Note that in [38], \mathcal{R}^m is denoted $A^{m,2}$ and e_j, μ_j are denoted $\frac{1}{2}\beta_j, 2m_j$.) \square

Theorem 5.4. *The eigenvalue of $C_{s,l}$ on $V_\mu \subset L^2(Y, p)$ is $q_l(2s - \rho_1 - p, \mu + \rho)$.*

Proof. The proof is very similar to Theorem 3.6 and we sketch the main arguments. We assume first that $p = b \geq 0$. As in Theorem 3.6 it suffices to prove the result for $l = 1$. By the Harish-Chandra homomorphism of Lemma 5.2 and the argument of Lemma 3.1, the eigenvalues $c_{s,\mu}$ of $C_s = C_{s,1}$ are given by a \mathcal{W}_r -invariant polynomial of degree $2r$ in $(s, \mu + \rho)$. We first prove that $c_{-m,\mu}$ vanishes if $2m = \mu_1 - b$.

For this we note that $C_{-m} = \Psi^{-m+1} L \Psi^m$ and hence

$$C_{-m} J_m = J_{m-1} L.$$

Now L maps the space $\mathcal{R}^{m+b,m}$ to $\mathcal{R}^{m-1+b,m-1}$ and if $\mu_1 = 2m + b$ then by Lemma 5.3 the K -type μ occurs in $\mathcal{R}^{m+b,m}$ but not in $\mathcal{R}^{m-1+b,m-1}$, therefore L acts by 0 on this K -type. Hence $C_{-m} J_m$ acts by 0, since J_m is an isomorphism on this K -type, C_{-m} acts by 0 on V_μ and so $c_{-m,\mu} = 0$.

Finally by \mathcal{W}_r -invariance of $c_{s,\mu}$ it follows that up to an overall constant multiple we have

$$(5.3) \quad c_{s,\mu} = \prod_{j=1}^r [(2s - \rho_1 - p)^2 - (\mu_j + \rho_j)^2] = q_1(2s - \rho_1 - p, \mu + \rho).$$

As before, equality follows by an easy calculation of leading coefficients.

For $p < 0$, we realize $C^\infty(Y, \pm p)$ as function spaces on X as above and consider the following operator T acting on functions on X

$$Tf(x) = \det(x^\dagger x)^p f(\bar{x}).$$

It is easy to see that T is a K -isomorphism from $C^\infty(Y, -p)$ to $C^\infty(Y, p)$. Moreover, the function $\Psi(x) = \det(x^\dagger x)$ and the operator $L = \partial(\Psi)$ are invariant under conjugation $x \mapsto \bar{x}$, and it follows that we have

$$TC_{s,l} = C_{s-p,l}T.$$

By the result for $p > 0$, the eigenvalue of $C_{s,l}$ on V_μ in $C^\infty(Y, -p)$ is

$$q_l(2(s - p) - \rho_1 - (-p), \mu + \rho) = q_l(2s - \rho_1 - p, \mu + \rho)$$

as claimed. \square

We also consider the non-compact dual \tilde{Y} of Y , and the space

$$(5.4) \quad C^\infty(\tilde{Y}, p) = \{f : Z \rightarrow \mathbb{C} \mid f(zg) = \det(g)^p f(z) \text{ for all } g \in G_r\}$$

where the open set $Z \subset W$ is as in (4.1), then the operators $C_{s,l}$ act as invariant differential operators on $C^\infty(\tilde{Y}, p)$. The p -spherical functions

$$\phi_{\eta,p}; \quad \eta \in \tilde{\mathfrak{a}}^*, p \in \mathbb{Z},$$

as defined in [35], belong to the space $C^\infty(\tilde{Y}, p)$ [38, Theorem 4.6], and are simultaneous eigenfunctions for the $C_{s,l}$. As in Theorem 4.1 one may compute the eigenvalues by writing η in the form $\tilde{\lambda}$ for $\lambda \in \mathfrak{a}^*$ as in (4.2).

Theorem 5.5. *For all $\lambda \in \mathfrak{a}^*$ we have $\tilde{C}_{s,l}(\phi_{\tilde{\lambda},p}) = q_l(2s - \rho_1 - p, \lambda + \rho)\phi_{\tilde{\lambda},p}$.*

Proof. This is proved by exactly the same method as Theorem 4.1 using Theorem 5.4. \square

Remark 5.6. For any p there is a $\tilde{K}_n = U(r, n - r)$ -invariant L^2 space of sections of the corresponding line bundle. For negative integer $p \leq -n$ the L^2 -space of holomorphic sections forms the holomorphic discrete series, and it is generated by the spherical function $\phi_{\lambda, p}$ at $\lambda = -\rho - (p, \dots, p)$; see [35]. In particular it is annihilated by C_0 which is closely related to Shimura operators [39]; see the remark below.

Remark 5.7. Let $n = 2r$ and $p = 0$. The symmetric space \tilde{Y} is the tube domain $U(n, n)/U(n) \times U(n)$. The space $C^\infty(\tilde{Y})$ is now realized as homogeneous functions on the subset Z of $2r \times r$ -matrices $z = [z_1, z_2]^t$ satisfying (5.4) for $p = 0$. Let now $s = -(r - 1)$. Our results above claim that the differential operator²

$$\tilde{C}_s = 2^{2r} \det(z_1^* z_1 - z_2^* z_2)^r \det(\partial_1^* \partial_1 - \partial_2^* \partial_2)^r \det(z_1^* z_1 - z_2^* z_2)^{1-r}$$

has eigenvalue

$$\prod_{j=1}^r (1 - (i\lambda_j)^2).$$

The operator \tilde{C}_s is up to a non-zero constant, the Shimura operator \mathcal{L}_{1^r} [36, 39], as can be seen by comparing \tilde{C}_s with the explicit formula for \mathcal{L}_{1^r} in the Siegel domain realization [39, formula (3.8)]. The eigenvalue for \tilde{C}_s is also a consequence of [39, Th. 3.4], noticing again here we are using different basis, our basis vectors $\{e_j\}$ are $\{\frac{1}{2}\beta_j\}$ there.

6. INVERTING THE RADON TRANSFORM

We shall apply our results above to find explicit differential operators inverting the Radon transform. To begin with, we fix integers r, r' such that

$$(6.1) \quad r \leq n - r, \quad r \leq r',$$

and we consider the corresponding Grassmannian manifolds

$$Y = Gr_{n,r}(\mathbb{F}) = K_n/K_r \times K_{n-r}, \quad Y' = Gr_{n,r'}(\mathbb{F}) = K_n/K_{r'} \times K_{n-r'}$$

We write y and η for typical elements of Y and Y' and define the incidence sets

$$A_\eta = \{y \in Y : y \subset \eta\} \text{ and } B_y = \{\eta \in Y' : y \subset \eta\}.$$

These are homogeneous spaces for the stabilizers K_η and K_y of η and y in $K = K_n$, and hence admit unique invariant probability measures dy and $d\eta$. The Radon transforms $R = R_{r',r} : C^\infty(Y) \rightarrow C^\infty(Y')$ and $R' = R_{r,r'} : C^\infty(Y') \rightarrow C^\infty(Y)$ are defined to be

$$(6.2) \quad Rf(\eta) = \int_{A_\eta} f(y) dy, \quad R'F(y) = \int_{B_y} F(\eta) d\eta.$$

Let V_μ be a K_n -isotypic subspace of $C^\infty(Y)$ as in Sect. 2.3. Since $R'R$ is a K_n -invariant operator on Y , by Schur's lemma there exist scalar eigenvalues $\gamma_\mu = \gamma_\mu(r, r', n)$ such that

$$(6.3) \quad R'Rv = \gamma_\mu v \text{ for all } v \in V_\mu,$$

²Here we use the usual convention for complex differentiation, $\partial_w = \frac{1}{2}(\partial_u - i\partial_v)$ for a complex variable $w = u + iv$. The appearance of the coefficient 2^{2r} is due to the fact that our isomorphism (2.3) is done through the underlying Euclidean inner product

We first describe an explicit formula for these eigenvalues γ_μ . For this we set

$$(6.4) \quad \alpha = \frac{d}{2}r, \beta = \frac{d}{2}(n - r'), l = \frac{d}{2}(r' - r);$$

let $(c)_m = c(c+1) \cdots (c+m-1)$ denote the Pochhammer symbol, and for μ define

$$m_i = \mu_i/2, \quad \mathbf{m} = (m_1, \dots, m_r), \quad |\mathbf{m}| = m_1 + \cdots + m_r,$$

$$(c)_{\mathbf{m}} = \prod_{j=0}^{r-1} (c - dj/2)_{m_{j+1}}, \quad \eta_\mu(\nu) = \frac{(-1)^{|\mathbf{m}|}(-\nu)_{\mathbf{m}}}{(dn/2 + \nu)_{\mathbf{m}}}.$$

Theorem 6.1. *For n, r, r' as in (6.1), we have.*

$$(6.5) \quad \gamma_\mu = \eta_\mu(-\alpha) \eta_\mu(-\beta) = \frac{(\alpha)_{\mathbf{m}}(\beta)_{\mathbf{m}}}{(\alpha + l)_{\mathbf{m}}(\beta + l)_{\mathbf{m}}}.$$

This result is proved in [8] for the complex case, and stated without proof for the real and quaternionic cases. While the real case can indeed be proved similarly, it seems to us that the quaternionic case requires substantially different arguments. Moreover the statement in [8, Theorem 6.2] for the quaternionic case is wrong. As an immediate test, the formula there cannot be correct because it vanishes on the constant function. To be more precise we note first that all the factors in the displayed formula should be inverted, and the factors should be

$$(m_j + 2(k+1) - 2j)(m_j + 2(k+1) - 2j + 1)(m_j + 2(n-k-1) - 2j)(m_j + 2(n-k-1) - 2j + 1).$$

The constant k, n and $m_j, j = 0, \dots, k$ there are our $r-1, n-1$ and $\frac{\mu_i}{2}, i = 1, \dots, r$ respectively; Theorem 6.2 [8] is stated for the Radon transform $R_{k,k+1}$ in the notation there, namely for our $R_{r+1,r}$ with $r' = r+1$ and $l = 2$.

For the above reasons we give a different calculation of γ_μ , using ideas from [40, 41], where the Radon transform is studied along with the spherical transform of the sine functions on Grassmannians and their non-compact duals, the real bounded symmetric domains. It was proved that when $r \leq \frac{r'}{2}$ the eigenvalues of the Radon transform are given by the evaluation of spherical polynomials at certain specific points up to a unknown factor; see Lemma 7.3 below. In the present paper we first find an explicit formula for this special value.

We follow the notation in [40] and let $\{E_j\}_{j=1}^r$ denote the basis of the Cartan subalgebra \mathfrak{a} dual to $\{e_j\}_{j=1}^r$. We write $K = K_n$ and $M = K_r \times K_{n-r}$ and consider two points y_0 and y_1 in the space $Y = K/M$. The point y_0 is the identity coset in Y as in Section 2, while y_1 is the coset of the element

$$\exp \frac{i\pi}{2} (E_1 + \cdots + E_r)$$

Let ϕ_μ be the M -spherical function in the space V_μ , i.e., the unique element M -invariant function normalized so that $\phi_\mu(y_0) = 1$. As in [40] let $|\sin y|$ be the M -invariant function on Y whose restriction to \mathfrak{a} is given by the sine function.

The following lemma is proved in [41] for the non-compact dual symmetric spaces of Y and Y' , the proof for the compact case is exactly the same. Let $\eta_0 \in Y'$ be the space $\eta_0 = [\mathbb{F}^{r'}]$ with $y_0 = [\mathbb{F}^r]$ the standard subspace in η_0 . Then the incidence set of η_0 is the symmetric subspace of Y

$$A_{\eta_0} = Gr_{r',r}(\mathbb{F}) = K_{r'}/K_r \times K_{r'-r} \subset Y$$

of rank $\min(r, r' - r)$. In particular if $r \leq \frac{r'}{2}$ it is of the same rank as Y with \mathfrak{a} as a corresponding Cartan subspace.

Lemma 6.2. *Let f be a M -invariant smooth function on Y .*

(1) *For n, r, r' as in (6.1) we have*

$$(6.6) \quad (R'Rf)(y_0) = Rf(\eta_0) = \int_{A_{\eta_0}} f(y) dy.$$

(2) *If furthermore $r \leq \frac{r'}{2}$ then this can be written as an integration on Y ,*

$$(6.7) \quad (R'Rf)(y_0) = Rf(\eta_0) = \int_Y |\sin y|^{-2\beta} f(y) dy,$$

where β is as in (6.4) and dy is the K -invariant measure and $|\sin y|^{-2\beta} dy$ is an M -invariant probability measure on Y .

We next need a couple of rationality results for $\phi_\mu(y_1)$ and γ_μ .

Lemma 6.3. *For fixed r and μ , $\phi_\mu(y_1)$ is a rational function of n .*

Proof. It is well known (see e.g. [18, Theorem 9.1]) that $\phi_\mu(\exp(it_1 E_1 + \cdots + it_r E_r))$ is a polynomial in $\sin t_1, \dots, \sin t_r$ of degree $2|\mu|$, with coefficients depending rationally on the root multiplicities, hence on n for fixed r . In particular the special value at y_1 depends rational on n . \square

Lemma 6.4. *For fixed r and μ , $\gamma_\mu(n, r, r')$ is a rational function of n and r' .*

Proof. For $r < \frac{r'}{2}$ we can write $\gamma_\mu(n, r, r')$ an integral over \mathfrak{a} . Explicitly we have

$$\begin{aligned} \gamma_\mu(n, r, r') &= \int_Y |\sin y|^{-2\beta} \phi_\mu(y) dy \\ &= \int_{Q_0} \phi_\mu(\exp(it_1 E_1 + \cdots + it_r E_r)) \prod_{\alpha \in R_+} |\sin \alpha(it)|^{m'_\alpha} dt_1 \cdots dt_r; \end{aligned}$$

see also [40, (2.5), (3.11)] with the integration being normalized as well. Here $m'_\alpha = m_\alpha - 2\beta = d(n - 2r) - d(n - r') = d(r' - 2r) > 0$ if α is the root e_j and $m'_\alpha = m_\alpha$ other wise, with m_α being the restricted root multiplicity specified in Section 3, and Q_0 is a fundamental polygon in \mathfrak{a} . Now there exist two orthogonal bases ϕ_μ and ϕ'_μ , which are the Heckman-Opdam polynomials [23], with respect to the root multiplicity function m_α and m'_α respectively, all normalized by $\phi(0) = 1$. The integral above is precisely the constant term in the expansion of ϕ_μ in terms of ϕ'_μ . But both ϕ_μ and ϕ'_μ have triangular expansions in terms of the monomials with coefficients that are rational in both m_α and m'_α respectively, hence rational in n, r' ; see [18, Theorem 9.1], which in turn implies that ϕ_μ can be expanded in terms of ϕ'_μ with rational coefficients. This proves the Lemma. \square

Thus in the proofs below we may assume, if necessary, that

$$(6.8) \quad n \gg r' \gg r.$$

Lemma 6.5. *We have $\gamma_\mu = \eta_\mu(-\beta) \phi_\mu(y_1)$.*

Proof. We may assume $r' > 2r$. By (7.2) and (7.6) we have $R'R\phi_\mu(y_0) = \gamma_\mu$ is as an integration of ϕ_μ on M against the function $|\text{Sin}|^{-2\beta}$, and it is shown in [40] that for any ν

$$(6.9) \quad \int_Y |\text{Sin } y|^{2\nu} \phi_\mu(y) dy = \eta_\mu(\nu) \phi_\mu(y_1)$$

The result now follows by applying (6.7) to ϕ_μ and using (6.9). \square

The value $\phi_\mu(y_1)$ was only found [40] for certain symmetric spaces. In the present paper we shall find an evaluation formula by using the interpretation in [40] of the Gelfand integral formula [11, Ch. IV, Sect. 2, Proposition 2.2] as a Radon transform. This formula might be also of independent interest. The reader may also consult [11, Ch. I, Sect. 4, Lemma 4.9] and [12, Ch. I, Sect. 3, Remark, pp.55-56] for the relevance of this formula in the study of Radon transform.

Proposition 6.6. *We have $\phi_\mu(y_1) = \eta_\mu(-\alpha)$, where α is as in Theorem 7.1.*

Proof. We may assume that $r \leq \frac{n}{3}$ and follow the proof of [40, Lemma 4.5], which shows that

$$(6.10) \quad \phi_\mu(y_1)^2 = R_{r,n-r} R_{n-r,r} \phi_\mu(y_0).$$

Now by the previous Lemmas 7.2 and 7.5, with r' replaced by $n-r$ and β by α , we get

$$R_{r,n-r} R_{n-r,r} \phi_\mu(y_0) = [\eta_\mu(-\alpha) \phi_\mu(y_1)] \phi_\mu(y_0) = \eta_\mu(-\alpha) \phi_\mu(y_1)$$

Combining these formulas we get

$$\phi_\mu(y_1)^2 = \eta_\mu(-\alpha) \phi_\mu(y_1),$$

and it suffices to show that $\phi_\mu(y_1) \neq 0$. However for $r \leq n/2$ the map $R_{r,n-r} R_{n-r,r}$ is injective, which follows for example from the analytical inversion formulae in [9, 42], and thus by (6.10) we must have $\phi_\mu(y_1) \neq 0$ \square

We now prove Theorem 6.1.

Proof. The identity $\gamma_\mu = \eta_\mu(-\alpha) \eta_\mu(-\beta)$ follows directly from Lemma 7.5 and Proposition 7.6. To rewrite this in the form $\frac{(\alpha)_m(\beta)_m}{(\alpha+l)_m(\beta+l)_m}$ we recall the definition of $\eta_\mu(-\nu)$ and use the identity $dn/2 = \alpha + \beta + l$. \square

We will use Theorem 6.1 to find an inverse for the Radon transform under the assumptions

$$(6.11) \quad r \leq r' \leq n-r;$$

and the following parity condition, which is automatic for $\mathbb{F} = \mathbb{C}, \mathbb{H}$,

$$(6.12) \quad l = \frac{d}{2}(r' - r) \text{ is an integer.}$$

We show that in this setting γ_μ^{-1} is a *symmetric polynomial* in $\bar{\mu} = \mu + \rho$. More precisely, we recall from Theorem 3.5 that the differential operator C_s has eigenvalues

$$c_{s,\mu} = \prod_{j=1}^r [(2s + \rho_1)^2 - (\mu_j + \rho_j)^2]$$

and we will relate the eigenvalue γ_μ to the product

$$\varepsilon_\mu = \prod_{i=0}^{l-1} c_{-\beta-i, \mu} = \prod_{j=1}^r \prod_{i=0}^{l-1} [(-2\beta - 2i + \rho_1)^2 - \bar{\mu}_j^2]$$

Lemma 6.7. *Under the parity assumption (6.12) we have $\gamma_\mu^{-1} = \varepsilon_\mu / \varepsilon_0$.*

Proof. If a_μ is a function of μ and S is a multiset of real numbers, we will write

$$a_\mu \sim S \text{ iff } a_\mu = c \prod_{j=1}^r \prod_{s \in S} (s - \bar{\mu}_j) \text{ for some constant } c \text{ independent of } \mu$$

Using $\rho_j - \rho_1 = d(j-1)$ and the identity $\frac{(c+l)_m}{(c)_m} = \frac{\Gamma(c+l+m)\Gamma(c)}{\Gamma(c+m)\Gamma(c+l)} = \frac{(c+m)_l}{(c)_l}$, we get

$$\frac{(\alpha+l)_m}{(\alpha)_m} = \prod_{j=1}^r \frac{(\alpha + l + \frac{\rho_j - \rho_1}{2})_{m_j}}{(\alpha + \frac{\rho_j - \rho_1}{2})_{m_j}} = \prod_{j=1}^r \frac{(\alpha + m_j + \frac{\rho_j - \rho_1}{2})_l}{(\alpha + \frac{\rho_j - \rho_1}{2})_l} = c \prod_{j=1}^r \left(\frac{2\alpha - \rho_1 + \bar{\mu}_j}{2} \right)_l.$$

Thus writing $\alpha_i = -2\alpha - 2i + \rho_1$ and $\beta_i = 2\beta + 2i - \rho_1$ we have

$$(6.13) \quad \frac{(\alpha+l)_m}{(\alpha)_m} \sim \{-\alpha_0, \dots, -\alpha_{l-1}\}, \quad \frac{(\beta+l)_m}{(\beta)_m} \sim \{-\beta_0, \dots, -\beta_{l-1}\}.$$

We now use the identity $\alpha + \beta = dn/2 - l$ to deduce that

$$\alpha_{i-1} + \beta_{l-i} = -2(\alpha + \beta) + 2\rho_1 - 2l + 2 = -dn + 2\rho_1 - 2 = 0.$$

Thus we have $-\alpha_{i-1} = \beta_{l-i}$ and now by (6.13) we get

$$\gamma_\mu^{-1} = \frac{(\alpha+l)_m}{(\alpha)_m} \frac{(\beta+l)_m}{(\beta)_m} \sim \{\pm\beta_0, \dots, \pm\beta_{l-1}\}.$$

It follows that there is a constant c such that

$$\gamma_\mu^{-1} = c \prod_{j=1}^r \prod_{i=0}^{l-1} (\beta_i^2 - \bar{\mu}_j^2) = c\varepsilon_\mu.$$

Since $\gamma_0 = 1$, we deduce $c = \varepsilon_0^{-1}$ and the result follows. \square

Let C_s be the differential operator from (2.10), and define

$$S = \varepsilon_0^{-1} \prod_{i=0}^{l-1} C_{-\beta-i} = \varepsilon_0^{-1} \Psi^{\beta+l} L^l \Psi^{-\beta}.$$

Theorem 6.8. *Under the assumptions (6.11) and (6.12) we have $SR'R = I$.*

Proof. It is enough to prove that $SR'R$ acts on each V_μ by 1. By Theorem 3.5 the operator S acts on $\mathcal{H}(\mu)$ by the scalar eigenvalue

$$(6.14) \quad \varepsilon_0^{-1} \prod_{i=0}^{l-1} c_{-\beta-i, \mu} = \varepsilon_0^{-1} \prod_{i=0}^{l-1} (\beta_i^2 - \bar{\mu}_j^2) = \varepsilon_\mu / \varepsilon_0 = \gamma_\mu^{-1}.$$

Now by (6.3) $SR'R$ acts on V_μ by $\gamma_\mu^{-1} \gamma_\mu = 1$. \square

Remark 6.9. (Relation to the Funk transform.) Let $V = V_{n,r}$ and $U = V_{n,k}$ be the Stiefel manifolds of orthonormal r -frames and k -frames in \mathbb{R}^n , and for $u \in U$ define

$$V^u := \{v \in V : u^t v = 0\}.$$

This is a homogeneous space for the stabilizer of u in $O(n)$ and hence carries a unique invariant probability measure $d_u v$. The Funk transform $F = F_{r,k}$ is defined as follows

$$F : C^\infty(V) \rightarrow C^\infty(U), \quad Ff(u) = \int_{V^u} f(v) d_u v.$$

In [26, Theorem 8.2] Rubin obtains an inversion formula for F restricted to $O(r)$ -invariant functions under the conditions

$$(6.15) \quad r \leq k \leq n - r, \quad n - k - r \text{ is even.}$$

We now explain how to deduce Rubin's result from ours. Since $V^u = V^{ur}$ for all $g \in O(k)$ the image of F is contained in $O(k)$ -invariant functions. Moreover we have natural identifications

$$C^\infty(V_{n,r})^{O(r)} \approx C^\infty(Gr_{n,r}), \quad C^\infty(V_{n,k})^{O(k)} \approx C^\infty(Gr_{n,k}) \approx C^\infty(Gr_{n,n-k})$$

where the last isomorphism corresponds to taking orthogonal complements. The restriction of F to $O(r)$ -invariant functions agrees with the Radon transform $R = R_{r',r}$ with $r' = n - k$. We note also that the conditions (6.15) are precisely the special case $d = 1$ of our conditions (1.7) for $r' = n - k$. The inversion formula of [26, Theorem 8.2] has the same form as ours, up to a certain explicit constant $c_{r,k}$, and we now give a brief argument to show that this is precisely $c_{\beta,l}(\rho)$ as in Theorem 1.2. For this we recall the Gindikin Gamma function [3] associated with the Jordan algebra A ,

$$\Gamma_A(x) = \prod_{j=1}^r \Gamma(x_j - \frac{d}{2}(j-1)), \quad x = (x_1, \dots, x_r).$$

For $\mathbf{m} = (m_1, \dots, m_r)$ we put $z + \mathbf{m} = (z + m_1, \dots, z + m_r)$; then an easy calculation shows that for a K -type $\mu = 2 \sum_{j=1}^r m_j e_j$ we have

$$c_{s,l}(\rho + \mu) = (-1)^{lr} 2^{2lr} \frac{\Gamma_A(s + l + \mathbf{m}) \Gamma_A(-s + \frac{dn}{2} + \mathbf{m})}{\Gamma_A(s + \mathbf{m}) \Gamma_A(-s + \frac{dn}{2} - l + \mathbf{m})}.$$

Specializing to $\mu = 0$ and $s = \beta = \frac{d}{2}(n - r')$ we get

$$c_{\beta,\lambda}(\rho) = (-1)^{-lr} 2^{-2lr} \frac{\Gamma_A(\frac{d}{2}(n - r')) \Gamma_A(\frac{d}{2}r)}{\Gamma_A(\frac{d}{2}(n - r)) \Gamma_A(\frac{d}{2}r')}.$$

For the real case $d = 1$ this is precisely the constant $c_{r,k}$ [26, Theorem 8.2].

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DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, BRUNSWICK, NJ, USA

E-mail address: `siddhartha.sahi@gmail.com`

MATHEMATICAL SCIENCES, CHALMERS UNIVERSITY OF TECHNOLOGY AND MATHEMATICAL SCIENCES, GÖTEBORG UNIVERSITY, SE-412 96 GÖTEBORG, SWEDEN

E-mail address: `genkai@chalmers.se`